

On Products of Fourier Coefficients of Cusp Forms

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Abstract

The purpose of this paper is to study products of Fourier coefficients of an elliptic cusp form, $a(n)a(n+r)$ ($n \geq 1$) for a fixed positive integer r , concerning both non-vanishing and non-negativity.

1 Introduction

Let f be an elliptic cusp form of positive integral weight k with real Fourier coefficients $a(n)$ ($n \geq 1$). Let r be a fixed positive integer. Then, the purpose of this paper is to study the products $a(n)a(n+r)$ ($n \geq 1$), regarding both non-vanishing and non-negativity. For a precise statement of our results see section 2 below.

We remark that these products have been previously investigated under different aspects, namely first by Selberg [6] and by Good [1] who studied growth properties of the sums $\sum_{n \leq x} a(n)a(n+r)$ where $x \rightarrow \infty$, and more recently by Hoffstein and Hulse [2] in connection with shifted Dirichlet convolutions.

2 Statement of results

Let \mathbb{H} denote the upper half-plane of complex numbers $z = x + iy$, with $y > 0$. The modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ acts on \mathbb{H} by fractional linear transformations, as usual.

In the following, we denote by Γ a discrete subgroup of Γ_1 , which satisfies the following conditions [cf. 1, p. 98]:

- (i) Γ is a finitely generated Fuchsian group of the first kind.
- (ii) The negative identity matrix is contained in Γ .
- (iii) Γ contains $M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ exactly if b is an integer.

Condition (iii) means that Γ has a cusp at $i\infty$, the stabilizer of which is generated by the two matrices $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We shall prove the following theorem

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Theorem 1. *Let f be a cusp form of integer weight $k > 2$ on Γ with real Fourier coefficients $a(n)$ ($n \geq 1$). Let $r \in \mathbb{N}$ and assume that $(a(n)a(n+r))_{n \geq 1}$ is not identically zero. Then, in fact, infinitely many terms of the sequence $(a(n)a(n+r))_{n \geq 1}$ are non-zero.*

The following corollary is an immediate consequence of the theorem. Recall only that for f a normalized Hecke eigenform, the Fourier coefficients are real and $a(1) = 1$.

Corollary 1. *Let f be a cusp form of integer weight $k > 2$ and level N that is a normalized Hecke eigenform with Fourier coefficients $a(n)$ ($n \geq 1$). Let $r \in \mathbb{N}$ and suppose that $a(r+1) \neq 0$. Then, the sequence $(a(n)a(n+r))_{n \geq 1}$ has infinitely many non-vanishing terms.*

From Corollary 1, we obtain

Corollary 2. *Let f be a normalized Hecke eigenform of integer weight k on Γ_1 with Fourier coefficients $a(n)$ ($n \geq 1$). Then, there are infinitely many n such that $a(n)$ and $a(n+1)$ are both non-zero.*

Proof. By results from [7] and [3], the second Fourier coefficient $a(2)$ is non-zero modulo any prime \mathfrak{p} lying above 5 (in an appropriate finite extension of \mathbb{Q}), hence is non-zero. The assertion follows. \square

The proof of Theorem 1 which will begin in section 3 makes use of a Dirichlet series associated to the sequence $(a(n)a(n+r))_{n \geq 1}$ and of its analytic properties. This series was introduced by Selberg [6] and later studied by Good [1], see above.

Further, assuming Γ is a congruence subgroup of level N , we use a result of Knopp, Kohnen and Pribitkin [4], Theorem 1, on sign changes of Fourier coefficients of cusp forms to show the following:

Theorem 2. *Let f be a cusp form of integer weight k on Γ , with real Fourier coefficients $a(n)$ ($n \geq 1$). Let $r \in \mathbb{N}$. Then the sequence $(a(n)a(n+r))_{n \geq 1}$ has infinitely many non-negative terms.*

Remark. *In the same way, one can prove that the sequence $(a(n)a(n+r))_{n \geq 1}$ has infinitely many non-positive terms.*

The proof of this theorem is carried out in section 4. We note that, in fact a somewhat more general statement holds, as the requirements of [4], which need to be considered in addition to our conditions (i)–(iii), are minimal (see the remark on p. 8 below).

Ideally, one might hope to prove a sign change result for the sequence $a(n)a(n+r)$ ($n \geq 1$) by combining Theorems 1 and 2 appropriately. However we have not been able to do this.

3 Proof of Theorem 1

We shall actually prove a more general statement than the assertion of Theorem 1.

We assume that the Fourier expansion of f is given by $f(z) = \sum_{n \geq 1} a(n)e^{2\pi i n z}$ with $a(n) \in \mathbb{R}$ ($\forall n \geq 1$). Let $r \in \mathbb{N}$ be fixed. For $n \geq 1$, set

$$c_n := a(n)a(n+r).$$

To the sequence $(c_n)_{n \in \mathbb{N}}$, we attach a Dirichlet series [see 1, 6] by setting

$$D(s, r) := \sum_{n \geq 1} \frac{c_n}{(n + \frac{r}{2})^s} \quad (\sigma := \operatorname{Re}(s) > k). \quad (1)$$

The theorem we shall prove is the following:

Theorem 1'. *Assume that $c_n \geq 0$ for almost all $n \geq 1$ and that the sum*

$$D(s, r) := \sum_{n \geq 1} c_n (n + \frac{r}{2})^{-s} \quad (\sigma > k)$$

in fact converges for all $s \in \mathbb{C}$. Then $c_n = 0$ for all $n \geq 1$.

We postpone the proof of this Theorem, and give the proof of Theorem 1 first.

Proof of Theorem 1. It suffices to assume that all but a finite number of the coefficients c_n are zero and to derive a contradiction.

Thus, let c_{n_1}, \dots, c_{n_p} , with $n_1 < \dots < n_p$ be those which are non-zero. Then, the Dirichlet series $D(s, r)$ from (1) becomes a Dirichlet polynomial

$$D(s, r) = \sum_{i=1}^p \frac{c_{n_i}}{(n_i + \frac{r}{2})^s},$$

and hence converges in the entire s -plane. Also, by hypothesis almost all of its coefficients are ≥ 0 . Now, by applying Theorem 1', we find that $c_n = 0$ for all $n \geq 1$. This is a contradiction, and the proof is complete. \square

The rest of this section is dedicated to the proof of Theorem 1'. First, we fix some notation: In the following, denote by $z = x + iy$ a complex variable with real part x and imaginary part y . By \mathcal{F} denote a fundamental domain for the action of Γ on \mathbb{H} , and by $L^2(\Gamma \backslash \mathbb{H})$ the Hilbert space of complex-valued Γ -invariant functions on \mathbb{H} that are square integrable on \mathcal{F} with respect to the invariant measure $d\nu = \frac{dx dy}{y^2}$. If f and g are in $L^2(\Gamma \backslash \mathbb{H})$ we denote the inner product by

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z) \overline{g(z)} d\nu(z).$$

Even if f and g do not both belong to $L^2(\Gamma \backslash \mathbb{H})$ we continue to use the notation for the above integral, as long as it converges absolutely. Finally, set $F(z) := y^k |f(z)|^2$.

Proof of Theorem 1'. We loosely follow the notation used by Good in [1]. Fix a maximal system of Γ -inequivalent cusps, $\xi_1 = \infty, \xi_2, \dots, \xi_\kappa$. (Note that by our assumptions on Γ , this system is finite and contains ∞ .)

For $r \in \mathbb{N}$, the Poincaré series $P_0(z, s, r)$ [see 1, p. 13] is defined as

$$P_0(z, s, r) = \frac{(\pi r)^{s-1/2}}{\Gamma(s + \frac{1}{2})} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s e(r \Re(\gamma z)),$$

for a complex variable $s = \sigma + it$ with $\sigma > 1$. Here, Γ_∞ stands for the stabilizer of ξ_1 in Γ , consisting of the matrices $M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{Z}$.

For each cusp ξ_i , denote by $g_i \in \Gamma_1$ an element with $g_i(\xi) = \infty$ and for which $\Gamma_{\xi_i} := g_i^{-1} \Gamma_\infty g_i$ is the stabilizer of ξ_i in Γ . Then, the non-holomorphic Eisenstein series of weight zero attached to the cusp ξ_i (cf. [1, pp. 104f] or e.g. [5, Chapter II]) is defined as

$$E_i(z, s) = \sum_{\gamma \in \Gamma_{\xi_i} \backslash \Gamma} (\Im(g_i^{-1} \gamma z))^s \quad (\sigma > 1).$$

This sum converges absolutely in the half-plane $\sigma > 1$. The Eisenstein series has meromorphic continuation to the entire s -plane. Its Fourier expansion is given by

$$E_i(z, s) = \delta_{i,1} y^s + \phi_i(s) y^{1-s} + \sum_{m \geq 1} \phi_i(m; s) 2y^{1/2} K_{s-1/2}(2\pi|m|y) e(mx),$$

with coefficient functions

$$\begin{aligned} \phi_i(s) &= \frac{\sqrt{\pi} \cdot \Gamma(s + \frac{1}{2})}{\Gamma(s)} L_i^{(0)}(s), & \phi_i(m; s) &= \frac{\pi^s |m|^{s-\frac{1}{2}}}{\Gamma(s)} L_i^{(m)}(s), \\ \text{where } L_i^{(m)} &= \sum_{c>0} c^{-2s} \sum_{d \bmod c} e\left(m \frac{d}{c}\right) & ((\begin{smallmatrix} * & \\ c & d \end{smallmatrix}) \in \Gamma_{\xi_i}). \end{aligned}$$

Note that, by the functional equation of $E_i(z, s)$, one has $\phi_i(-m; 1-s) = \overline{\phi_i(m, s)}$.

Now, consider $\langle P_0(\cdot, s, r), F \rangle$ for $\sigma > 1$. As a function in s , it has holomorphic continuation to a small neighborhood of the line $\sigma = \frac{1}{2}$ and there, the functional equation holds [cf. 1, Lemma 5, p. 115f]:

$$\begin{aligned} \langle P_0(\cdot, s, r), F \rangle &= \frac{1}{2s-1} \sum_{i=1}^{\kappa} \phi_i(-r; 1-s) \langle E_i(\cdot, s), F \rangle + \langle P_0(\cdot, s-1, r), F \rangle \\ &\quad + (4\pi)^{\frac{1}{2}-k} \left(\frac{n}{4}\right)^{s-\frac{1}{2}} \frac{\Gamma(k+s-1)}{\Gamma(s+\frac{1}{2})} \sum_{n \geq 1} \frac{c_n}{(n+\frac{r}{2})^{k+s-1}} \Delta_r(s, n), \end{aligned} \tag{2}$$

where $\Delta_r(s, n)$ is given by [see 1, p. 119],

$$\begin{aligned} \Delta_r(s, n) := & 1 - F\left(\frac{k+s-1}{2}, \frac{k+s}{2}, s + \frac{1}{2}; \left(\frac{r}{2n+r}\right)^2\right) \\ & - \frac{\Gamma(k-s)\Gamma(s-\frac{1}{2})}{\Gamma(k+s-1)\Gamma(\frac{1}{2}-s)} \left(\frac{4n+2r}{r}\right)^{2s-1} \times \\ & \times \left[F\left(\frac{k-s}{2}, \frac{k-s+1}{2}, \frac{3}{2}-s; \left(\frac{r}{2n+r}\right)^2\right) - 1 \right]. \end{aligned} \quad (3)$$

Here, $F(a, b, c; z)$ is the hypergeometric function

$$F(a, b, c; z) = \sum_{w=0}^{\infty} \frac{(a)_w (b)_w}{(c)_w w!} z^w \quad (|z| < 1), \quad (a)_w = \frac{\Gamma(a+w)}{\Gamma(a)}.$$

From the expression in (3) one sees immediately that the series on the right hand side of (2) is convergent at least for $-1 < \sigma < 2$.

Also, note that the Dirichlet series satisfies [cf. 1, Lemma 5 (i), p. 116]

$$D(s+k-1, r) = (4\pi)^{k-\frac{1}{2}} \left(\frac{4}{r}\right)^{s-1/2} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(k+s-1)} \langle P_0(\cdot, s, r), F \rangle \quad (\sigma > 1). \quad (4)$$

Since by hypothesis, $D(s, r)$ converges for all $s \in \mathbb{C}$, we obtain from (2), (3) and (4) by continuation, that the identity

$$\begin{aligned} (4\pi)^{\frac{1}{2}-k} \left(\frac{r}{4}\right)^{s-\frac{1}{2}} \frac{\Gamma(k+s-1)}{\Gamma(s+\frac{1}{2})} D(k+s-1, r) = \\ \frac{1}{2s-1} \sum_{i=1}^{\kappa} \phi_i(-r; 1-s) \langle E_i(\cdot, s), F \rangle \\ + (4\pi)^{\frac{1}{2}-k} \left(\frac{r}{4}\right)^{\frac{1}{2}-s} \frac{\Gamma(k-s)}{\Gamma(\frac{3}{2}-s)} \cdot D(k-s, r) \\ + (4\pi)^{\frac{1}{2}-k} \left(\frac{r}{4}\right)^{s-\frac{1}{2}} \frac{\Gamma(k+s-1)}{\Gamma(s+\frac{1}{2})} \sum_{n \geq 1} \frac{c_n}{(n+\frac{r}{2})^{k+s-1}} \cdot \Delta_r(s, n) \end{aligned} \quad (5)$$

holds between meromorphic functions for all $s \in \mathbb{C}$.

Now, take residues at $s = k + 2m$ ($m \in \mathbb{N}_0$) on both sides of (5). By [5, Theorems 4.3.4, 4.3.5, p. 43], the Eisenstein series $E_i(z, s)$ are holomorphic in $\sigma > \frac{1}{2}$ except for finitely many poles in the interval $(\frac{1}{2}, 1]$, which are precisely the poles of the constant coefficients $\phi_i(s)$. It follows that the $\phi_i(r; s)$ are holomorphic at $s = k + 2m$ and do not contribute to the residue.

Thus, we pick up non-zero residues only from the second and third terms on the right-hand side of (5); in the third term, the only non-zero contribution comes from the second (non-constant) term of $\Delta_r(s, n)$ in (3). Note that $\text{res}_{s=-m} \Gamma(s) = \frac{(-1)^m}{m!}$ for $m \in \mathbb{N}_0$.

More precisely, we have

$$\begin{aligned}
0 &= \left(\frac{r}{4}\right)^{\frac{1}{2}-k-2m} \frac{1}{\Gamma(\frac{3}{2}-k-2m)} \frac{1}{(2m)!} \cdot D(-2m, r) \\
&- \left(\frac{r}{4}\right)^{k+2m-\frac{1}{2}} \frac{\Gamma(2k+2m-1)}{\Gamma(k+2m+\frac{1}{2})} \sum_{n \geq 1} \frac{c_n}{(n+\frac{r}{2})^{2k+2m-1}} \frac{(2mn)^{-1} \cdot \Gamma(k+2m-\frac{1}{2})}{\Gamma(2k+2m-1) \cdot \Gamma(\frac{1}{2}-k-2m)} \times \\
&\quad \times \left(\frac{4n+2r}{r}\right)^{2k+4m-1} \left[F\left(-m, -m+\frac{1}{2}, \frac{3}{2}-k-2m, \left(\frac{r}{2n+r}\right)^2\right) - 1 \right].
\end{aligned}$$

We note that

$$\Gamma\left(\frac{3}{2}-k-2m\right) = \left(\frac{1}{2}-k-2m\right) \cdot \Gamma\left(\frac{1}{2}-k-2m\right)$$

and

$$\Gamma\left(k+2m+\frac{1}{2}\right) = \left(k+2m-\frac{1}{2}\right) \cdot \Gamma\left(k+2m-\frac{1}{2}\right).$$

Thus, we obtain

$$\begin{aligned}
0 &= \left(\frac{r}{4}\right)^{\frac{1}{2}-k-2m} \frac{1}{(\frac{1}{2}-k-2m)} D(-2m, r) \\
&- \left(\frac{r}{4}\right)^{k+2m-\frac{1}{2}} \frac{1}{(k+2m-\frac{1}{2})} \sum_{n \geq 1} \frac{c_n}{(n+\frac{r}{2})^{2k+2m-1}} \times \\
&\quad \times \left(\frac{4n+2r}{r}\right)^{2k+4m-1} \left[F\left(-m, -m+\frac{1}{2}, \frac{3}{2}-k-2m; \left(\frac{r}{2n+r}\right)^2\right) - 1 \right].
\end{aligned}$$

Whence, further

$$\begin{aligned}
0 &= \left(\frac{r}{4}\right)^{\frac{1}{2}-k-2m} D(-2m, r) + \left(\frac{r}{4}\right)^{k+2m-\frac{1}{2}} \sum_{n \geq 1} \frac{c_n}{(n+\frac{r}{2})^{2k+2m-1}} \times \\
&\quad \times \left(\frac{4n+2r}{r}\right)^{2k+2m-1} \left[F\left(-m, -m+\frac{1}{2}, \frac{3}{2}-k-2m; \left(\frac{r}{2n+r}\right)^2\right) - 1 \right].
\end{aligned}$$

Also, we have

$$\frac{1}{(n+\frac{r}{2})^{2k+2m-1}} \cdot \left(\frac{4n+2r}{r}\right)^{2k+4m-1} = \left(\frac{4}{r}\right)^{2k+4m-1} \left(n+\frac{r}{2}\right)^{2m}.$$

Therefore altogether, we obtain

$$D(-2m, r) = \sum_{n \geq 1} c_n \left(n+\frac{r}{2}\right)^{2m} \cdot \left[-1 + F\left(-m, -m+\frac{1}{2}, \frac{3}{2}-k-2m; \left(\frac{r}{2n+r}\right)^2\right) \right],$$

for all $r \in \mathbb{N}_0$. It follows that for all r

$$0 = \sum_{n \geq 1} c_n (2n + r)^{2m} F \left(-m, -m + \frac{1}{2}, \frac{3}{2} - k - 2m; \left(\frac{r}{2n + r} \right)^2 \right).$$

By the definition of the hypergeometric function, we have

$$F(-m, -m + \frac{1}{2}, \frac{3}{2} - k - 2m; z) = \sum_{w \geq 0} \frac{(-m) \cdots (-m + w - 1)(-m + \frac{1}{2}) \cdots (-m + w - 1 + \frac{1}{2})}{(-k - 2m + \frac{3}{2}) \cdots (-k - 2m + \frac{3}{2} + w - 1)} \frac{z^w}{w!}.$$

This is a polynomial in z of degree m , which we denote as $P_m(z)$. All of its coefficients $\lambda_{w,m}$, ($0 \leq w \leq m$) are non-zero. Set $\alpha_{\nu,m} = r^{2\nu} \lambda_{\nu,m}$ ($0 \leq \nu \leq m$) and write $P_m(z) = \sum_{\nu=0}^m \alpha_{\nu,m} z^{2m-2\nu}$. We find that

$$\sum_{n \geq 1} c_n P_m(2n + r) = 0.$$

Since all the $\alpha_{\nu,m}$ are non-zero, we find immediately that for all $\nu \in \mathbb{N}_0$,

$$\sum_{n \geq 1} c_n (2n + r)^{2\nu} = 0. \quad (6)$$

Now recall that (by hypothesis) almost all the coefficients c_n are ≥ 0 . Trivially, if there are non negative coefficients, since the right hand side is zero, there can be no positive coefficients, either and so $c_n = 0$ for all n , in which case we are finished. Thus, we assume that a finite number of coefficients, c_{n_1}, \dots, c_{n_t} , with $n_1 < n_2 < \dots < n_t$ are negative. For all $\nu \in \mathbb{N}_0$, we can write the equation (6) in the form

$$\sum_{\substack{n \geq 1 \\ n \neq n_1, \dots, n_t}} c_n (2n + r)^{2\nu} = -c_{n_1} (2n_1 + r)^{2\nu} - \dots - c_{n_t} (2n_t + r)^{2\nu}$$

Whence for all ν ,

$$\sum_{\substack{n \geq 1 \\ n \neq n_1, \dots, n_t}} c_n \left(\frac{2n + r}{2n_t + n} \right)^{2\nu} = -c_{n_1} \left(\frac{2n_1 + r}{2n_t + r} \right)^{2\nu} - \dots - c_{n_t}.$$

Here, as ν grows arbitrarily large, the right-hand side is positive and bounded, whereas, on the left-hand side, there exist n with $n > n_t$ (and with $c_n > 0$), thus, the left-hand side becomes arbitrarily large. This is a contradiction, hence $c_n = 0$ for all $n > n_t$.

Now, from (6) we get the system of linear equations with $0 \leq \nu \leq n_t - 1$

$$\sum_{n=1}^{n_t} c_n (2n + r)^{2\nu} = 0.$$

the determinant of which is easily seen to be non-zero. It follows that the remaining coefficients c_1, \dots, c_{n_t} vanish, too. Thus, to conclude, $c_n = 0$ for all $n \geq 1$. \square

4 Proof of Theorem 2

Recall that for this theorem, we require Γ to be a congruence subgroup of level N (then, condition (i) from p. 1 is satisfied automatically). For the proof, we need the following lemma.

Lemma 1. *Let f be cusp form with respect to Γ , with Fourier coefficients $a(n)$. Let $n_0 \geq 1$. Then, the Fourier expansion*

$$g(z) := \sum_{\substack{n \geq 1 \\ n \equiv n_0 \pmod{2r}}} a(n) e^{2\pi i n z}$$

defines a cusp form for a congruence subgroup of higher level.

Proof. With the usual $|_k$ -operation, we can write g in the form

$$\begin{aligned} g(z) &= \frac{1}{\phi(2r)} \sum_{s \pmod{2r}} e^{-\pi i \frac{n_0 s}{r} z} f \Big|_k \begin{pmatrix} 1 & \frac{s}{2r} \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{\phi(2r)} \sum_{n \geq n_0} a(n) e^{2\pi i n z} \sum_{s \pmod{2r}} e^{2\pi i \frac{(n-n_0)s}{2r}} \end{aligned}$$

Here, ϕ denotes the totient function, and summation runs over a minimal system of representatives $(\pmod{2r})$. Clearly, the right-hand side is modular with level a multiple of $4N$, and is cuspidal if f is. \square

Now, we are ready to prove Theorem 2.

Proof. Set $c(n) = a(n)a(n+r)$ for all $n \geq 1$. Assume that $c(n) \geq 0$ holds for only finitely many n . Then, there is an index $n_0 \in \mathbb{N}$, with the property that $c(n) < 0$ for all $n \geq n_0$.

Since $c(n_0) = a(n_0)a(n_0+r) < 0$ it follows that $a(n_0)$ and $a(n_0+r)$ are both non-zero and have opposite sign. By induction, it follows that all coefficients $a(n)$ with $n = n_0 + 2rl$ ($l \in \mathbb{N}_0$) have the same sign. Now, by Lemma 1, the resulting sequence of coefficients $(a(n))_{n \equiv n_0(2r)}$ defines a cusp form

$$g(z) = \sum_{\substack{n \geq 1 \\ n \equiv n_0 \pmod{2r}}} a(n) e^{2\pi i n z}$$

with real Fourier coefficients, of which, by construction only a finite number are less than or equal to zero. However, by the results of Knopp, Kohnen and Pribitkin [4], Theorem 1, if all Fourier coefficients of a cusp form are real, the sequence of its coefficients has infinitely many terms of either sign. This is a contradiction. \square

Remark. *Actually, for the result from [4] we use here, the only requirement is that the group mentioned in Lemma 1, besides being a Fuchsian group of the first kind with finite covolume, have $i\infty$ and 0 as parabolic fixed points. However, if Γ already satisfies conditions (i)–(iii) (see p. 1), this imposes only a mild further restriction. Thus, Theorem 2 holds somewhat more generally than only for congruence subgroups.*

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